UNIQUENESS OF BALANCED METRICS ON HOLOMORPHIC VECTOR BUNDLES

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ABSTRACT. Let $E \to M$ be a holomorphic vector bundle over a compact Kähler manifold (M, ω). We prove that if E admits a ω-balanced metric (in X. Wang's terminology [18]) then it is unique. This result together with [2] implies the existence and uniqueness of ω-balanced metrics of certain direct sums of irreducible homogeneous vector bundles over rational homogeneous varieties. We finally apply our result to show the rigidity of ω-balanced Kähler maps into Grassmannians.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let $E \to M$ be a very ample holomorphic vector bundle over a compact Kähler manifold (M, ω) and let $\underline{s} = (s_1, \dots, s_N)$ be a basis of $H^0(M, E)$, the space of global holomorphic sections of E. Let $i_{\underline{s}} : M \to G(r, N)$, $r = \operatorname{rank} E$, be the Kodaira map associated to the basis $\underline{s}$ (see, e.g. [13]), namely the holomorphic embedding whose expression $i_{\underline{s}} : U \to G(r, N)$ in a local frame $(\sigma_1, \dots, \sigma_r) : U \to E$ is given by:

$$i_{\underline{s}}(x) = \begin{bmatrix} S_{11}(x) & \dots & S_{1r}(x) \\ \vdots & & \vdots \\ S_{N1}(x) & \dots & S_{Nr}(x) \end{bmatrix}, \ x \in U, \tag{1}$$

where $s_j = \sum_{\alpha=1}^r S_{j\alpha}\sigma_\alpha$, $j = 1, \dots, N$. Here the square bracket denotes the equivalence class in $G(r, N) = M^*(r, N, \mathbb{C})/GL(r, \mathbb{C})$, where $M^*(r, N, \mathbb{C})$ is the set of $r \times N$ complex matrices of rank r.

Consider the flat metric h_0 on the tautological bundle $\mathcal{T} \to G(r, N)$, i.e. $h_0(v, w) = w^* v$, and the dual metric $h_{Gr} = h_0^*$ on the quotient bundle $\mathcal{Q} = \mathcal{T}^*$. Hence, we can endow $E = i_{\underline{s}}^* \mathcal{Q}$ with the hermitian metric

$$h_{\underline{s}} = i_{\underline{s}}^* h_{Gr} \tag{2}$$

and define a L^2-product on $H^0(M, E)$ by the formula:

$$< \cdot, \cdot >_{h_{\underline{s}}} = \frac{1}{V(M)} \int_M h_{\underline{s}}(\cdot, \cdot) \frac{\omega^n}{n!}, \tag{3}$$

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where $\omega^n = \omega \wedge \cdots \wedge \omega$ and $V(M) = \int_M \frac{\omega^n}{n!}$.

An hermitian metric h on a very ample holomorphic vector bundle $E \to M$ over a compact Kähler manifold (M, ω) is called *ω-balanced* if there exists a basis $\underline{s}$ of $H^0(M, E)$ such that $h = h_{\underline{s}} = i_{\underline{s}}^* h_{Gr}$ and

$$< s_j, s_k >_{h_{\underline{s}}} = \frac{r}{N} \delta_{jk}, j, k = 1, \ldots, N = \dim H^0(M, E). \tag{4}$$

One also says that $\underline{s}$ is a *ω-balanced* basis of $H^0(M, E)$ if (4) is satisfied. Therefore a metric on E is ω-balanced if it is the pull-back of the canonical metric h_{Gr} of $\mathcal{Q} \to G(r, N)$ via the Kodaira map associated to a ω-balanced basis $\underline{s}$ of $H^0(M, E)$. We refer the reader to [6] for the description of interesting conditions on a metric h of a holomorphic vector bundle $E \to M$ for the existence of a (not necesseraly balanced) basis $\underline{s}$ of $H^0(M, E)$ such that $h = h_{\underline{s}} = i_{\underline{s}}^* h_{Gr}$.

The concept of balanced metrics on complex vector bundles was introduced by X. Wang [18] (see also [19]) following S. Donaldson's ideas [9]. It can be also defined in the non-compact case and the study of balanced metrics is a very fruitful area of research both from mathematical and physical point of view (see, e.g., [4], [5], [8], [11], [10], [15] and [16]).

In [18] X. Wang proves that, under the assumption that the Kähler form ω is integral, E is Gieseker stable if and only if $E \otimes L^k$ admits a unique ω-balanced metric (for every k sufficiently large), where $L \to M$ is a polarization of (M, ω), i.e. L is a holomorphic line bundle over M such that $c_1(L) = [\omega]_{dR}$.

On the other hand, in Lemma 2.7 of [17], R. Seyyedali shows that if a simple bundle E (i.e. $\mathrm{Aut}(E) = \mathbb{C}^* \mathrm{id}_E$, where $\mathrm{Aut}(E)$ denotes the group of invertible holomorphic bundle maps from E in itself) admits a balanced metric then it is unique. In the following theorem, which is the main result of the present paper, we prove the unicity of balanced metrics for *any* vector bundle.

Theorem 1. *Let E be a holomorphic vector bundle over a compact Kähler manifold (M, ω). If E admits a ω-balanced metric then it is unique.*

As an application of Theorem 1 and L. Biliotti and A. Ghigi results [2] we obtain the existence and uniqueness of ω-balanced metrics over certain direct sum of homogeneous vector bundles over rational homogeneous varieties:

Theorem 2. *Let (M, ω) be a rational homogeneous variety and $E_j \to M, j = 1, \ldots, m$, be irreducible homogeneous vector bundles over M with $\mathrm{rank}\, E_j = r_j$ and $\dim H^0(M, E_j) = N_j > 0$, $j = 1, \ldots, m$. If $\frac{r_j}{N_j} = \frac{r_k}{N_k}$ for all $j, k = 1, \ldots, m$, then the homogeneous vector bundle $E = \oplus_{j=1}^m E_j \to M$ admits a unique homogeneous ω-balanced metric.*

Proof. Since $\mathrm{rank}\, E = \sum_{j=1}^m r_j$ and $\dim(H^0(M, E)) = \sum_{j=1}^m N_j$, it is enough to prove the theorem for $m = 2$. In [2] it is proved that each E_j, as in

the statement, is a very ample bundle and admits a unique homogeneous ω-balanced metric $\underline{s}^j = (s_1^j, \ldots, s_{N_j}^j)$, $j = 1, 2$. Then, the assumption $\frac{r_1}{N_1} = \frac{r_1}{N_2}$, readily implies that the basis

$$\underline{s} = ((s_1^1, 0), \ldots, (s_{N_1}^1, 0), (0, s_1^2), \ldots, (0, s_{N_2}^2)) \tag{5}$$

is a homogeneous ω-balanced basis for $E_1 \oplus E_2$. Then $h_{\underline{s}} = i_{\underline{s}}{}^* h_{Gr}$ is the desired homogeneous balanced metric on $E_1 \oplus E_2$ which is unique by Theorem 1. $\square$

The proof of Theorem 1 is based on Wang's work on balanced metrics (see [18] or the next section) and on moment map techniques developed by C. Arezzo and the first author in [1], where it is proved the unicity of *balanced metrics* in the sense of S. Donaldson [9]. Wang's work is summarized in the next section where we prove Lemma 1 which is fundamental for the proof of Theorem 1, to whom Section 3 is dedicated. In the last section we prove the rigidity of ω-balanced Kähler maps into Grassmannians.

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2. Balanced bases and moment map

Let E be a very ample holomorphic vector bundle over a compact Kähler manifold (M, ω). Let J_0 be the (complex) structure of E, denote by E_c the smooth complex vector bundle underlying E and write $E = (E_c, J_0)$. Let N be the complex dimension of $H^0(M, E)$ and let $\mathcal{H}$ be the (infinite dimensional) manifold consisting of pairs $(\underline{s}, J)$ where $\underline{s} = (s_1, \ldots, s_N)$ is an N-uple of complex linearly independent smooth sections of E_c, J is a complex structure of E_c and each section s_j is holomorphic with respect to the complex structure J, i.e.

$$ds_j \circ I_0 = J \circ ds_j, \ j = 1, \ldots, N,$$

where I_0 denotes the (fixed) complex structure of M.

Given an hermitian metric h on E we denote by $U_h(E_c)$ the subgroup of $GL(E_c)$ consisting of smooth invertible bundle maps $E_c \to E_c$ preserving the hermitian metric h and by $SU(N) \subset U(N)$ the group of $N \times N$ unitary matrices with positive determinant. These groups act in a natural way on $\mathcal{H}$ as follows:

$$\Psi \cdot (\underline{s}, J) = (\Psi \underline{s}, \Psi \cdot J), \ \Psi \in U_h(E_c)$$

$$U \cdot (\underline{s}, J) = (U \underline{s}, J), \ U \in SU(N),$$

were $\Psi \underline{s} = (\Psi s_1, \ldots, \Psi s_N)$, $\Psi \cdot J = \Psi J \Psi^{-1}$ and $U \underline{s} = (U s_1, \ldots, U s_N)$.

Since these actions commute they induce a well-defined action of the group $\mathcal{G}_h = U_h(E_c) \times SU(N)$ on $\mathcal{H}$. The Lie algebra of $\mathcal{G}_h$ is $GL(E_c) \oplus \mathfrak{su}(N)$ and its complexification $\mathcal{G}_h^{\mathbb{C}} = GL(E_c) \times SL(N)$ naturally acts on $\mathcal{H}$ by extending the action of $\mathcal{G}_h$.

Theorem 3 (Wang [18])**.** *The manifold $\mathcal{H}$ admits a Kähler form Ω invariant for the action of $\mathcal{G}_h$ whose moment map $\mu_h : \mathcal{H} \to GL(E_c) \oplus \mathfrak{su}(N)$ is given by:*

$$\mu_h(\underline{s}, J) = (\sum_{j=1}^{N} h(\cdot, s_j)s_j, \ < s_j, s_k >_h - \frac{\sum_{j=1}^{N} |s_j|^2_h}{N} \delta_{jk} \ ), \tag{6}$$

where $|s_j|^2_h = < s_j, s_j >_h = \frac{1}{V(M)} \int_M h(\cdot, \cdot) \frac{\omega^n}{n!}$. Consequently, a basis $\underline{s} = (\underline{s}, J_0)$ of $H^0(M, E)$ is balanced if and only if $\mu_{h_{\underline{s}}}(\underline{s}, J_0) = (\mathrm{id}_E, 0)$, where $h_{\underline{s}}$ is the metric of E given by (2).

A key ingredient in the proof of Theorem 1 is the following:

Lemma 1. *Let $\underline{s} = (\underline{s}, J_0)$ be a balanced basis of $H^0(M, E)$ and let $(\hat{\underline{s}}, \hat{J}) \in \mathcal{H}$ such that: $\mu_{h_{\underline{s}}}(\hat{\underline{s}}, \hat{J}) = (\mathrm{id}_E, 0)$ and $(\hat{\underline{s}}, \hat{J})$ lies in the same $\mathcal{G}^{\mathbb{C}}_{h_{\underline{s}}}$-orbit of $(\underline{s}, J_0)$. Then $(\hat{\underline{s}}, \hat{J})$ lies in the same $\mathcal{G}_{h_{\underline{s}}}$-orbit of $(\underline{s}, J_0)$, namely there exists $(\Psi, U) \in \mathcal{G}_{h_{\underline{s}}}$ such that $(\Psi, U) \cdot (\hat{\underline{s}}, \hat{J}) = (U\Psi\hat{\underline{s}}, \Psi \cdot \hat{J}) = (\underline{s}, J_0)$.*

Proof. Since $a = (\mathrm{id}_E, 0) \in GL(E_c) \oplus \mathfrak{su}(N)$ is (obviously) invariant by the coadjoint action of $\mathcal{G}_{h_{\underline{s}}}$, it is a standard fact in moment map's theory (see [14] for a proof and also Proposition 3.1 in [1]) that

$$\mu^{-1}_{h_{\underline{s}}}(a) \cap (\mathcal{G}^{\mathbb{C}}_{h_{\underline{s}}} \cdot x) = \mathcal{G}_{h_{\underline{s}}} \cdot x, \ \forall x \in \mu^{-1}_{h_{\underline{s}}}(a).$$

Then the result follows by the assumptions and by Theorem 3. □

3. The proof of Theorem 1

Let E be a very ample holomorphic vector bundle over a compact Kähler manifold (M, ω). If $\underline{s}$ is *any* basis of $H^0(M, E)$, $F \in \mathrm{Aut}(E)$ and $U \in U(N)$, then $i_{UF\underline{s}} = Ui_{F\underline{s}} = Ui_{\underline{s}}$ where $UF\underline{s} = (UFs_1, \dots, UFs_N)$ and it follows easily that $h_{\underline{s}} = h_{UF\underline{s}}$. Then the proof of Theorem 1 will be a consequence of the following:

Theorem 4. *If s and $\tilde{s}$ are two balanced bases of $H^0(M, E)$ then there exist a unitary matrix $U \in U(N)$ and $F \in \mathrm{Aut}(E)$ such that $\tilde{\underline{s}} = UF\underline{s}$.*

Proof. Let $h_{\underline{s}}$ and $h_{\tilde{\underline{s}}}$ be the metric induced by s and $\tilde{s}$ and $\Phi \in GL(E_c)$ such that $\Phi^* h_{\underline{s}} = h_{\tilde{\underline{s}}}$. We claim that

$$\sum_{j=1}^{N} h_{\underline{s}}(\cdot, \Phi\tilde{s}_j)\Phi\tilde{s}_j = \mathrm{id}_E \tag{7}$$

and

$$< \Phi\tilde{s}_j, \Phi\tilde{s}_k >_{h_{\underline{s}}} = \frac{r}{N}\delta_{jk}, \ j, k = 1, \dots, N. \tag{8}$$

Indeed

$$\mathrm{id}_E = \sum_{j=1}^{N} h_{\tilde{\underline{s}}}(\cdot, \tilde{s}_j)\tilde{s}_j = \sum_{j=1}^{N} (\Phi^* h_{\underline{s}})(\cdot, \tilde{s}_j)\tilde{s}_j$$

and if $\underline{\sigma} = (\sigma_1, \dots, \sigma_r) : U \to E$ is a local frame then, for all $\alpha = 1, \dots, r$, one gets:

$$\begin{aligned}\sigma_\alpha = \Phi(\Phi^{-1}(\sigma_\alpha)) &= \Phi\left(\sum_{j=1}^N h_{\underline{\tilde{s}}}(\Phi^{-1}(\sigma_\alpha), \tilde{s}_j)\tilde{s}_j\right)\\ &= \Phi\left(\sum_{j=1}^N (\Phi^* h_{\underline{s}})(\Phi^{-1}(\sigma_\alpha), \tilde{s}_j)\tilde{s}_j\right)\\ &= \Phi\left(\sum_{j=1}^N h_{\underline{s}}(\sigma_\alpha, \Phi\tilde{s}_j)\tilde{s}_j\right) = \sum_{j=1}^N h_{\underline{s}}(\sigma_\alpha, \Phi\tilde{s}_j)\Phi\tilde{s}_j,\end{aligned}$$

where we have used the fact that $\sum_{j=1}^N h_{\underline{\tilde{s}}}(\cdot, \tilde{s}_j)\tilde{s}_j = \mathrm{id}_E$, and (7) follows.

Moreover,

$$\begin{aligned}< \Phi\tilde{s}_j, \Phi\tilde{s}_k >_{h_{\underline{s}}} &= \frac{1}{V(M)}\int_M h_{\underline{s}}(\Phi\tilde{s}_j, \Phi\tilde{s}_k)\frac{\omega^n}{n!}\\ &= \frac{1}{V(M)}\int_M (\Phi^* h_{\underline{s}})(\tilde{s}_j, \tilde{s}_k)\frac{\omega^n}{n!}\\ &= \frac{1}{V(M)}\int_M h_{\underline{\tilde{s}}}(\tilde{s}_j, \tilde{s}_k)\frac{\omega^n}{n!} = \frac{r}{N}\delta_{jk}\end{aligned}$$

and also (8) is proved.

It follows by (6), (7) and (8) that $(\underline{s}, J_0)$ and $(\Phi\underline{\tilde{s}}, \Phi \cdot J_0)$ are in the same level set of the moment map $\mu_{h_{\underline{s}}}$, namely

$$\mu_{h_{\underline{s}}}(\underline{s}, J_0) = \mu_{h_{\underline{s}}}(\Phi\underline{\tilde{s}}, \Phi \cdot J_0) = (\mathrm{id}_E, 0).$$

Moreover, since $\underline{s}$ and $\underline{\tilde{s}}$ are bases of the same vector space $H^0(M, E)$ there exist a non zero constant λ and $V \in \mathrm{SL}(N)$ such that $\lambda V\underline{\tilde{s}} = \underline{s}$. Therefore

$$(\underline{s}, J_0) = (\lambda\Phi^{-1}, V) \cdot (\Phi\underline{\tilde{s}}, \Phi \cdot J_0)$$

and hence $(\underline{s}, J_0)$ and $(\Phi\underline{\tilde{s}}, \Phi J_0)$ are elements of $\mathcal{H}$ in the same $\mathcal{G}^{\mathbb{C}}_{h_{\underline{s}}}$-orbit. By Lemma 1 there exists $(\Psi, U) \in \mathcal{G}_{h_{\underline{s}}}$ such that

$$(\underline{s}, J_0) = (\Psi, U) \cdot (\Phi\underline{\tilde{s}}, \Phi \cdot J_0) = (U\Psi\Phi\underline{\tilde{s}}, (\Psi\Phi) \cdot J_0).$$

Consequently, $F = \Psi\Phi : E_c \to E_c$ preserves the complex structure J_0, i.e. $F \in \mathrm{Aut}(E)$ and $\underline{s} = UF\underline{\tilde{s}}$. □

4. Rigidity of ω-balanced Kähler maps into Grassmannians

Let (M, ω) be a compact Kähler manifold. A holomorphic map $f : M \to G(r, N)$ is said to be ω-balanced if there exist a very ample holomorphic vector bundle $E \to M$ and a balanced basis $\underline{s}$ of $H^0(M, E)$ such that $f = i_{\underline{s}}$ (thus necessarily $f^*\mathcal{Q} = E$, $r = \mathrm{rank}\, E$ and $N = \dim H^0(M, E)$). A ω-balanced map $f : M \to G(r, N)$ is called a *Kähler* map if $f^*\omega_{Gr} = \omega$, where ω_{Gr} is the standard Kähler form on $G(r, N)$, i.e. $\mathrm{Ric}(h_{Gr}) = \omega_{Gr}$.

Example 2. Let $M = \mathbb{C}P^1$ and $\omega_\lambda = \lambda\omega_{FS}$, where ω_{FS} is the Fubini-Study Kähler form and λ is a positive real number. Then, it not hard to see that the holomorphic map

$$f : \mathbb{C}P^1 \to G(2,4) : [z_0, z_1] \mapsto \begin{bmatrix} z_0 & 0 \\ 0 & z_0 \\ z_1 & 0 \\ 0 & z_1 \end{bmatrix} \tag{9}$$

is a ω_λ-balanced map for all λ. Moreover, f is Kähler when $\lambda = 2$, i.e. $f^*\omega_{Gr} = 2\omega_{FS}$. (In general it follows by (3) and (4) that if $\underline{s}$ is a ω-balanced basis of $H^0(M, E)$ then $\underline{s}$ is still $\lambda\omega$-balanced for $\lambda > 0$).

Note that in the previous example $f^*\mathcal{Q} = O(1) \oplus O(1)$, where $O(1)$ is the hyperplane bundle on $\mathbb{C}P^1$, $\mathrm{Ric}(h_{\underline{s}}) = 2\omega_{FS}$ (where $h_{\underline{s}} = f^*h_{Gr}$) and $f^*\omega_{Gr} = 2\omega_{FS}$. On the other hand, there exist holomorphic maps $\tilde{f} = i_{\underline{\tilde{s}}} : \mathbb{C}P^1 \to G(2,4)$ (where $\underline{\tilde{s}}$ is a basis of $H^0\left(\mathbb{C}P^1, O\left(1\right) \oplus O\left(1\right)\right)$ satisfying these three conditions but for which it cannot exist a unitary transformation U of $G(2,4)$ such that $\tilde{f} = Uf$ (cfr. [7]). An example is given by:

$$\tilde{f} : \mathbb{C}P^1 \to G(2,4) : [z_0, z_1] \mapsto \begin{bmatrix} {z_0}^2 & z_0\overline{z}_1\frac{1}{2}\left(\sqrt{3}-1\right) \\ -z_0z_1\frac{1}{2}\left(\sqrt{3}-1\right) & |z_0|^2 + \frac{1}{2}|z_1|^2\sqrt{3} \\ -z_0z_1\frac{1}{2}\left(\sqrt{3}+1\right) & -\frac{1}{2}|z_1|^2 \\ {z_1}^2 & \overline{z}_0z_1\frac{1}{2}\left(1-\sqrt{3}\right) \end{bmatrix}.$$

This phenomenon is due to the fact that the rigidity of Kähler maps into $G(r, N)$ with $r \geq 2$ does not in general hold true (see, e.g., [3], [7], [12], [20], [21]), in contrast with the case $r = 1$ where one has the celebrated Calabi's rigidity theorem for Kähler maps into projective spaces.

On the other hand the following theorem, which is the main result of this section, shows the rigidity of ω-balanced Kähler embedding.

Theorem 5. *Let $E \to M$ be a very ample complex vector bundle over a compact Kähler manifold (M, ω). Assume that E admits a ω-balanced metric h such that $\mathrm{Ric}(h) = \omega$. Then there exists a unique (up to a unitary transformations of $G(r, N)$) ω-balanced Kähler embedding $f : M \to G(r, N)$ such that $f^*Q = E$.*

Proof. Let $\underline{s}$ be a balanced basis of $H^0(M, E)$ and let $f = i_{\underline{s}} : M \to G(r, N)$ be the associated Kodaira's map. By Theorem 1 f is the unique (up to a unitary transformations of $G(r, N)$) ω-balanced embedding such that $f^*Q = E$. So it remains to show that $f^*\omega_{Gr} = \omega$. Fix a local frame $(\sigma_1, \ldots, \sigma_r) : U \to E$. In this local frame $f : U \to G(r, N)$ is given by (1). Then, the local expression of $h = h_{\underline{s}}$, $\omega = \mathrm{Ric}(h)$ and $f^*\omega_{Gr}$ are given respectively by $(S^*S)^{-1}$, $-\frac{i}{2}\partial\bar{\partial}\log\det(S^*S)^{-1}$ and $\frac{i}{2}\partial\bar{\partial}\log\det(S^*S)$. □

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